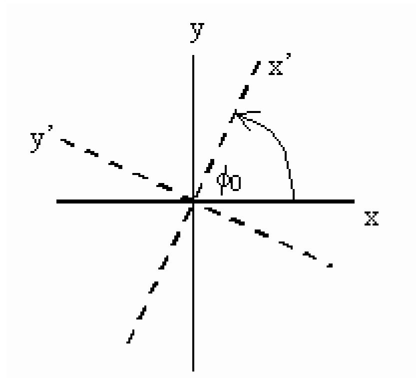
Details of the procedure:

First we find the coordinates of the point where the photon enters the sphere. To do this, we rotate our coordinate system ccw about the z-axis by an angle ϕ_0.



The photon is now incident in the x'z'-plane. It has momentum components

$$p_{z'} = -p\cos\theta_0, \quad p_{x'} = -p\sin\theta_0.$$

The center of the sphere is located at $\mathbf{r}_0' = (x_0', y_0', z_0')$, where

$$\begin{pmatrix} x_0' \\ y_0' \\ z_0' \end{pmatrix} = \begin{pmatrix} \cos\phi_0 & \sin\phi_0 & 0 \\ -\sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}.$$

The photon will contact the sphere at a point located in the x'-z' plane. The equation for the photon's trajectory is $z' = (1/\tan\theta_0)x'$. The equation for the surface of the sphere is $(x'-x_0')^2 + (y'-y_0')^2 + (z'-z_0')^2 = R^2$. The surface of the sphere intersects the x'z'-plane at points which satisfy the equation $(x'-x_0')^2 + (z'-z_0')^2 = R^2-y_0'^2$. This is the equation of a circle. If $R^2 < y_0'^2$ then there exists no intersection and the photon's momentum is unchanged.

Assume that the surface of the sphere intersects the x'z'-plane. If the line giving the photon's trajectory passes through the circle the we can find the point where the photon enters the sphere by inserting $z' = (1/\tan\theta_0)x'$ into the equation of the circle. This yields

$$(z'\tan\theta_0-x_0')^2 + (z'-z_0')^2 = R^2-y_0'^2,$$

or $Az'^2 + Bz' + C = 0$, with

$A = (1 + \tan^2(\theta_0)) = 1/\cos^2(\theta_0)$,
$B = -2(x_0'\tan\theta_0 + z_0')$,
$C = x_0'^2 + y_0'^2 + z_0'^2 - R^2$.

The constant A is equal to or larger than 1. Therefore the larger value of z' is given by

$$z_1 = \left(\frac{1}{2A}\right)\left(\sqrt{B^2 - 4AC} - B\right).$$

A solution exists if B^2-4AC is positive. Then $x_1 = \tan\theta_0 z_1$.

Assume now that in the primed coordinate system the photon enters the sphere at the point (x_1, 0, z_1). We will now shift the center of this coordinate system to the origin of the sphere. This shift does not change the orientation of the axes, and the momentum components of the photon are the same before and after the shift. The coordinates of the point where the photon enters the sphere become

$$(x_2, y_2, z_2) = (x_1 - x_0', -y_0', z_1 - z_0').$$

The vector pointing from the center of the sphere to (x_2, y_2, z_2) makes an angle θ_2 with the new z-axis and an angle ϕ_2 with the new x-axis, where

$$\sin\theta_2 = \frac{\sqrt{x_2^2 + y_2^2}}{R}, \quad \cos\theta_2 = \frac{z_2}{R},$$

$$\cos\phi_2 = \frac{x_2}{\sqrt{x_2^2 + y_2^2}}, \quad \sin\phi_2 = \frac{y_2}{\sqrt{x_2^2 + y_2^2}}.$$

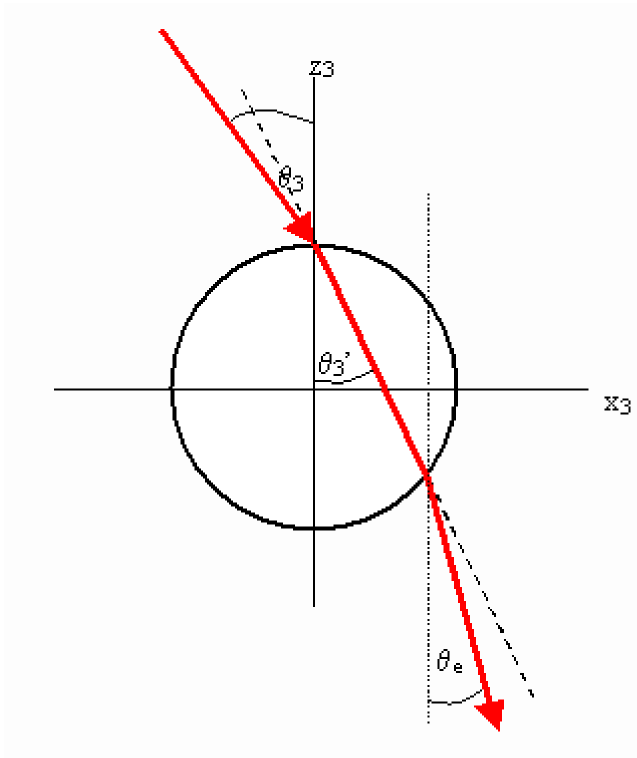
We now rotate this new coordinate system ccw about the z-axis through an angle ϕ_2. This moves the intersection point into the x-z plane of the rotated coordinate system, which we will label with superscript 1. The components of the photons momentum in this coordinate system become

$$\begin{pmatrix} p_x^{\ 1} \\ p_y^{\ 1} \\ p_z^{\ 1} \end{pmatrix} = \begin{pmatrix} \cos\phi_2 & \sin\phi_2 & 0 \\ -\sin\phi_2 & \cos\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x' \\ p_y' \\ p_z' \end{pmatrix}.$$

To move the intersection point onto the z-axis, we will now rotate the coordinate system ccw about the y-axis by an angle θ_2. We will denote the new coordinate system with a double prime. The components of the photon's momentum in this new coordinate system become

$$\begin{pmatrix} p_z'' \\ p_x'' \\ p_y'' \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_z^{\ 1} \\ p_x^{\ 1} \\ p_y^{\ 1} \end{pmatrix}.$$

Let $\tan\phi_3 = p_y''/p_x''$. One more rotation ccw about the z''-axis through an angle ϕ_3 yields the situation shown in the figure below with $\theta_3 = \tan^{-1}\left(\frac{\sqrt{p_x''^2 + p_y''^2}}{|p_z|}\right)$.



In this figure $\theta_e = 2\theta_3' - \theta_3$, where θ_3' is found from Snell's law, $n_1\sin\theta_3 = n_2\sin\theta_3'$. The momentum components of the photon leaving the sphere in the double primed coordinate system are

$$p_{zf}'' = -p\cos\theta_e, \quad p_{xf}'' = p\sin\theta_e\cos\phi_3, \quad p_{yf}'' = p\sin\theta_e\sin\phi_3,$$

with

$$\cos\phi_3 = \frac{p_x''}{\sqrt{p_x''^2 + p_y''^2}} \text{ and } \sin\phi_3 = \frac{p_y''}{\sqrt{p_x''^2 + p_y''^2}}.$$

To find the momentum vector of the photon in the original unprimed coordinate system the rotations of the coordinate axes have to be reversed.

$$\begin{pmatrix} p_{zf}^{1} \\ p_{xf}^{1} \\ p_{yf}^{1} \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_{zf}'' \\ p_{xf}'' \\ p_{yf}'' \end{pmatrix},$$

$$\begin{pmatrix} p_{xf}' \\ p_{yf}' \\ p_{zf}' \end{pmatrix} = \begin{pmatrix} \cos\phi_2 & -\sin\phi_2 & 0 \\ \sin\phi_2 & \cos\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_{xf}^{1} \\ p_{yf}^{1} \\ p_{zf}^{1} \end{pmatrix},$$

with

$$\sin\theta_2 = \frac{\sqrt{x_2^2 + y_2^2}}{R}, \quad \cos\theta_2 = \frac{z_2}{R},$$

$$\cos\phi_2 = \frac{x_2}{\sqrt{x_2^2 + y_2^2}}, \quad \sin\phi_2 = \frac{y_2}{\sqrt{x_2^2 + y_2^2}}.$$

Finally,

$$\begin{pmatrix} p_{xf} \\ p_{yf} \\ p_{zf} \end{pmatrix} = \begin{pmatrix} \cos\phi_0 & -\sin\phi_0 & 0 \\ \sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_{xf}' \\ p_{yf}' \\ p_{zf}' \end{pmatrix}.$$

The momentum transferred to the sphere is **P** =(P_x,P_y,P_z) = (p_x-p_{xf}, p_y-p_{yf}, p_z-p_{zf}).

To find the total momentum transferred to the sphere by the laser beam we have to choose an intensity profile for the laser beam. The simplest assumption is a uniform angular distribution of the photons in a beam with circular cross section, i.e. every solid angle element $\Delta\cos\theta\Delta\phi$ contains the same number of photons up to the maximum angle θ_{max}. We can now calculate **P** for many photons in the distributions and find the average **P** per refracted photon. The intensity of the laser beam yields the number of photons per second n. The force on the sphere is given by **F** = n**P**.